Sharp Corner Singularity of the White–Metzner Model

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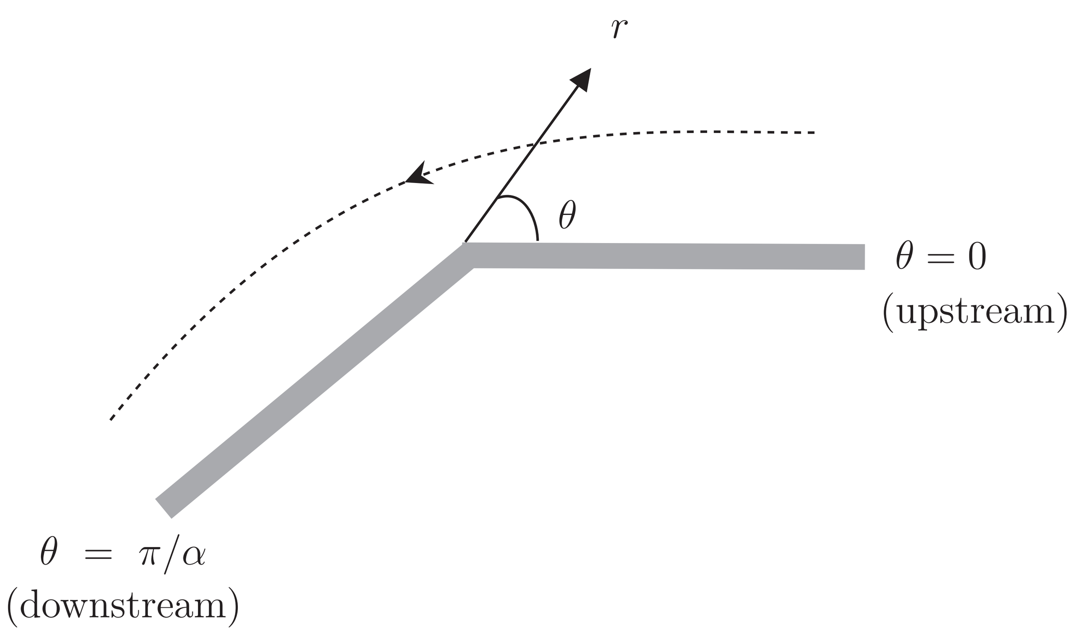
Introduction

Re-entrant corner flows are commonly found in polymer processing applications, for example in cross-slot and contraction flows. Here, we utilise matched asymptotic expansions to provide a description of White–Metzner (WM) fluid flow around a re-entrant corner. The WM model here assumes no solvent viscosity, with power-law variations in relaxation time and polymer viscosity. Compared to the Upper Convected Maxwell (UCM) model, WM shows the same singularity and boundary layer structure, however layer thickness varies depending on the relative difference in the exponents.

Geometry

We consider steady, incompressible, planar flow as below, with $1/2 \leq \alpha < 1$.



No-slip and no-flux conditions are applied to both walls.

Governing Equations

$$\mathrm{Re}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nabla\cdot\boldsymbol{T},$$

$$\boldsymbol{T} + \underbrace{\mathrm{Wi}\,\dot{\gamma}^{q-1}}_{\text{elasticity}}\overset{\nabla}{\boldsymbol{T}} = \underbrace{2\dot{\gamma}^{n-1}}_{\text{viscosity}}\boldsymbol{D}.$$

Shear rate: $\dot{\gamma} := \sqrt{2\boldsymbol{D}:\boldsymbol{D}}$.

Natural Stress Decomposition

Cartesian stress components fail to provide a complete leading order description of the corner flow. We instead write the conformation tensor $\boldsymbol{A} := \boldsymbol{T} + \dot{\gamma}^{n-q}\boldsymbol{I}$ as

$$\boldsymbol{A} = T_{uu}\mathbf{u}\mathbf{u}^T + T_{uw}\left(\mathbf{u}\mathbf{w}^T + \mathbf{w}\mathbf{u}^T\right) + T_{ww}\mathbf{w}\mathbf{w}^T.$$

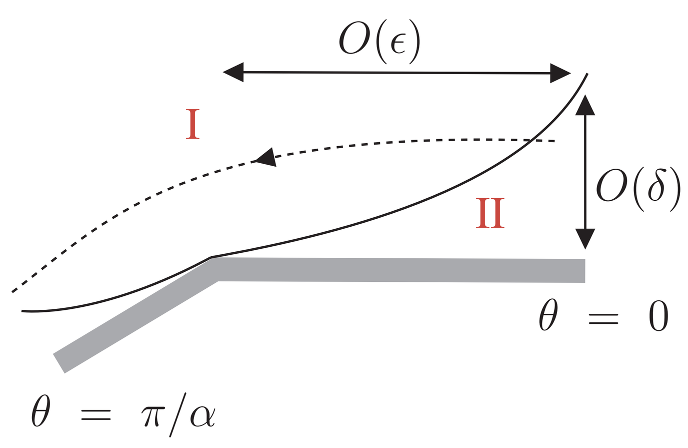
Asymptotic Structure

As $r \to 0$, we obtain a symmetric three region structure around the re-entrant corner:

1. Region I: Outer Solution

Close to corner ($r \ll 1$) but away from the walls, the fluid behaves elastically. The extra-stress is given by a stretching solution $\boldsymbol{T} \sim T_{uu}(\psi)\mathbf{u}\mathbf{u}^T$, with a 'potential flow' solution for the streamfunction ψ:

$$\psi \sim \frac{C_0}{\alpha^m} r^{m\alpha}\sin^m(\alpha\theta),\ \ \boldsymbol{T} = O\left(r^{2(\alpha-1)}\right),\ \ T_{uu,uw,ww} \sim d_i\left(\frac{\psi}{C_0}\right)^{m_i}.$$

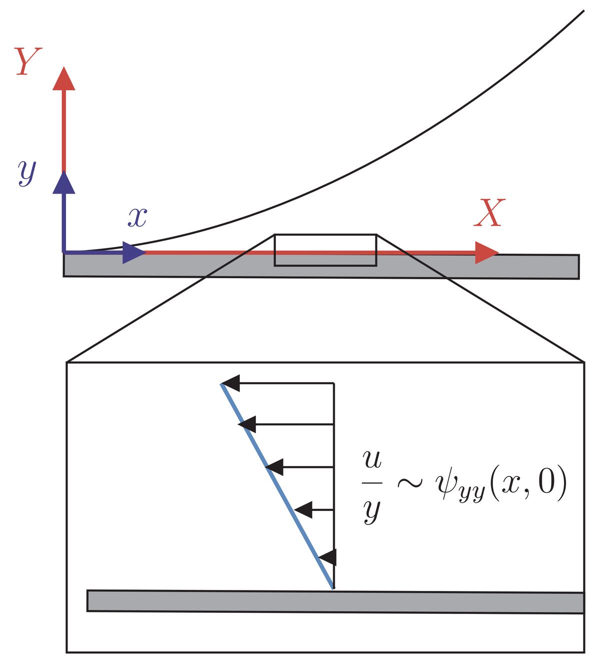


The powers m, m_i are known through boundary layer matching; C_0 and d_i are determined through solving the boundary layer equations.

2. Region II: Boundary Layers

The outer solution cannot capture viscometric behaviour near to the walls. Scaling the WM equations with $x = \epsilon X$, $y = \delta(\epsilon)Y$ shows existence of viscous boundary layers near both walls of thickness

$$\delta = \epsilon^{2-\alpha+\frac{(q-n)(1-\alpha)}{n+q}}.$$



For $q = n$, boundary layers are identical to the UCM model. For $q \neq n$, the boundary layer is thicker for $q < n$, and vice versa.

3. Similarity Solution

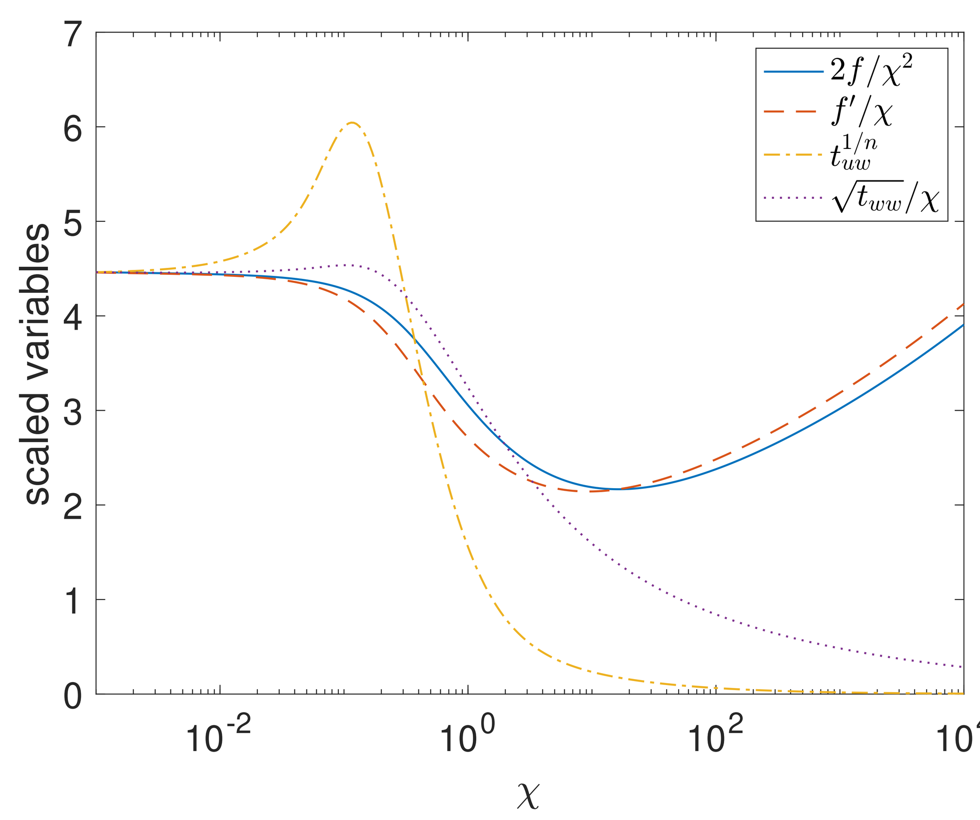
The boundary layer PDEs in $(\psi, T_{uu}, T_{uw}, T_{ww})$ can be converted to ODEs in $(f, t_{uu}, t_{uw}, t_{ww})$ via a similarity solution in

$$\chi = X^{-a}Y, \quad a = \frac{n + (3-2\alpha)q}{n+q}.$$

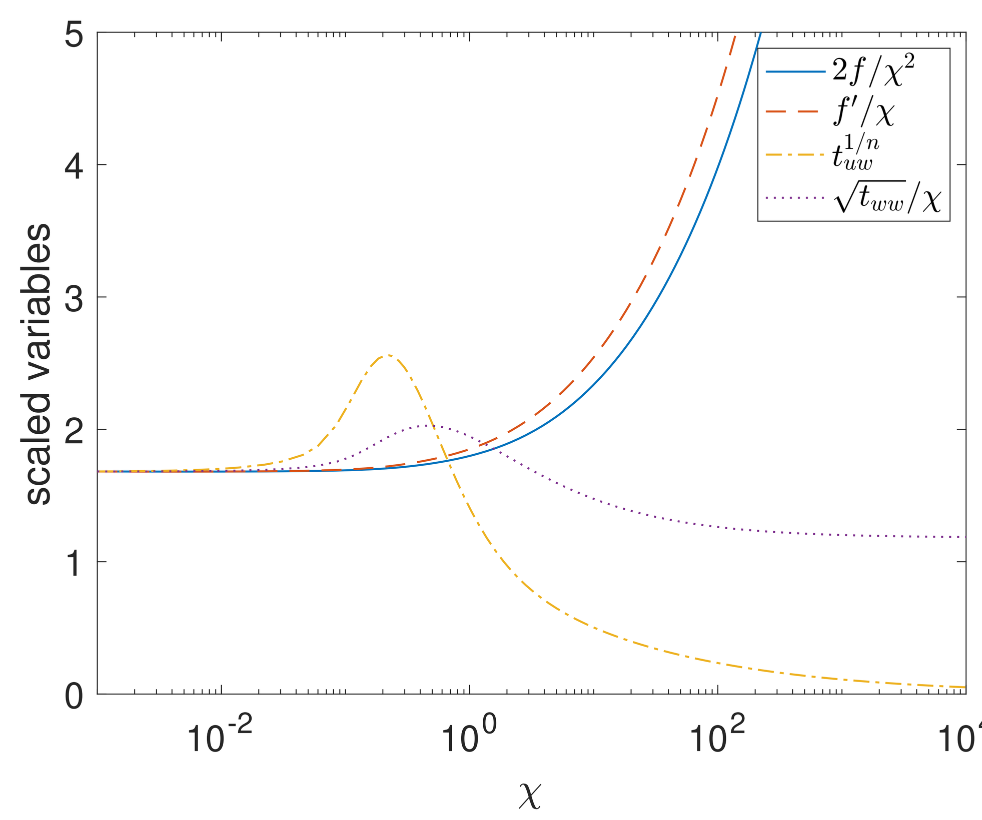
By imposing upstream pressure and wall shear rate values, the upstream stresses — and thus C_0 and d_i — can be determined via an initial value problem; these data can be used to solve for the downstream flow via a boundary value problem.

Downstream Solution and Shear Rate Estimation

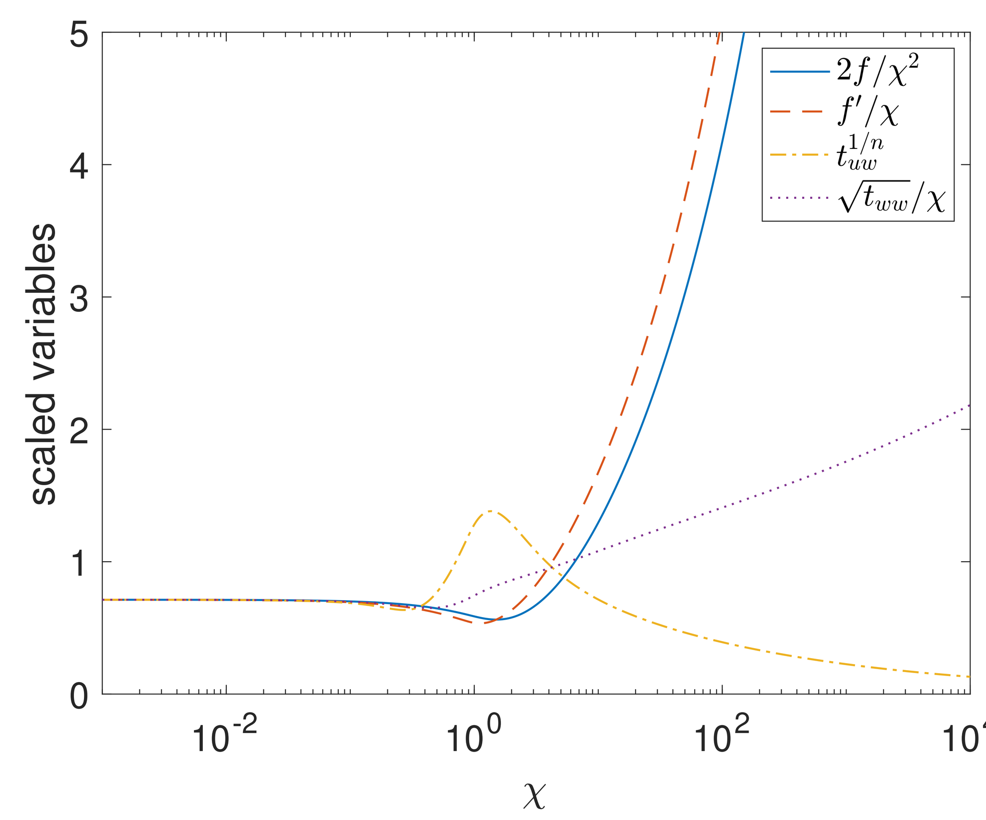
The boundary conditions mean near to the walls, $f \sim \frac{1}{2}f_2\chi^2$. By solving the similarity equations, we can estimate the downstream wall shear rate f_{2d}. We find that relative to UCM, shear thinning increases the downstream wall shear rate, and shear thickening decreases it.



(a) $n = 0.6$ (shear thinning)



(b) $n = 1$ (UCM — no shear thinning)



(c) $n = 1.4$ (shear thickening)

Here, $n = q$, $\mathrm{Wi} = 1$, $\alpha = 2/3$, the upstream wall shear rate is $f_{2u} = -1$ and the upstream pressure is $p = x^{2(\alpha-1)}$.

Future Work

1. Verify asymptotic results via full numerical simulation.
2. Investigate stress singularities in similar systems or models.

References

[1] J.D. Evans and C.A. Jones. "Sharp corner singularity of the White–Metzner model". Z. Angew. Math. Phys. 75 82(2024)

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