Re-Entrant Corner Flow of an Upper Convected Maxwell Fluid

TFR Transfer Talk

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Motivation

- Many materials we encounter have viscoelastic properties.
  - Viscosity: measure of resistance to flow.
  - Elasticity: ability of a material to return to its original state after removal of deforming forces.
- Due to polymer processing applications, modelling is incredibly important!

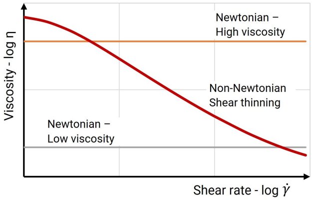
Sesame Workshop, 2017.

Cabrera, X.,2020

Kempner, J., 2022

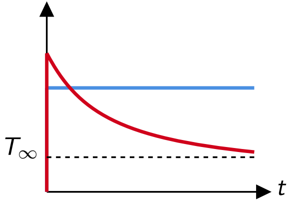
Properties of Viscoelastic Fluids

Shear Thinning:

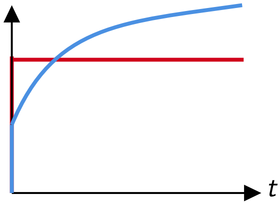


Østergård, A. L., 2020.

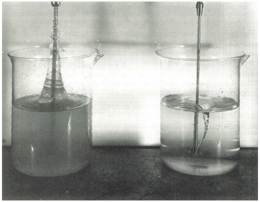
Stress Relaxation:



Creep:



Normal Stress Effects:



Tanner, R. I., 2000

Viscoelastic Modelling

Overview

Q. How can we model these materials?

A. Use conservation laws.

1. Mass:

$$\nabla \cdot \mathbf{u} = 0.$$

2. Momentum:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla\cdot\boldsymbol{T}, \quad \boldsymbol{T}^T = \boldsymbol{T}.$$

We still need to specify $\boldsymbol{T}$!

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Could try $\boldsymbol{T} = \eta \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) =: 2\eta \boldsymbol{D}$...

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Could try $\boldsymbol{T} = \eta \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) =: 2\eta \boldsymbol{D}$... but this doesn't help us!

Viscoelastic Modelling

Constitutive Equations

Can we do better?

Viscoelastic Modelling

Constitutive Equations

Can we do better?

- Acierno et. al.
- Chang et. al.
- Chilcott-Rallison
- Corotational Jeffreys
- Corotational Maxwell
- Curtiss-Bird
- Doi-Edwards
- Elastic Dumbbell
- Extended Pom-Pom
- Extended WM
- FENE
- FENE-P
- Giesekus
- Green-Tobolsky
- Jeffreys
- Johnson-Segelman
- Kaye-BKZ
- Kelvin-Voigt
- Leonov
- Lodge
- Maxwell
- Maxwell-Wiechert
- Modified UCM
- Oldroyd 8 Constant
- Oldroyd-B
- Phan-Thien-Tanner
- Pom-Pom
- Rolie-Poly
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- Upper Convected Maxwell
- White-Metzner
- Yamamoto

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Upper Convected Maxwell (UCM) Model

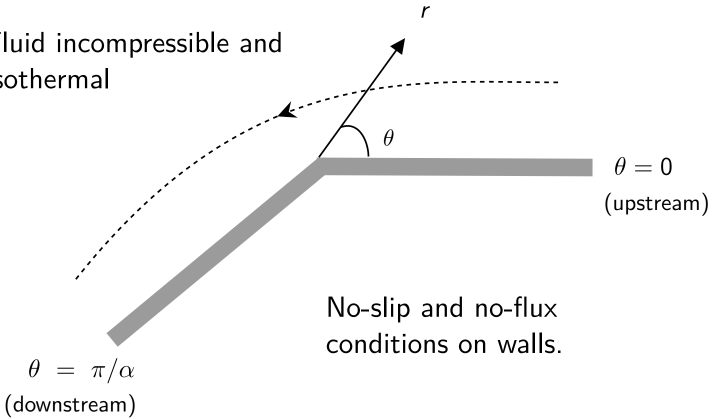
$$\tau \underbrace{\left(\frac{\partial \boldsymbol{T}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{T} - (\nabla \mathbf{u}) \, \boldsymbol{T} - \boldsymbol{T} \, (\nabla \mathbf{u})^{T} \right)}_{\overset{\nabla}{\boldsymbol{T}}} + \boldsymbol{T} = 2\eta \boldsymbol{D}$$

- First term captures elastic behaviour; remaining terms capture viscous behaviour.
- Introduced a *relaxation time* τ and viscosity parameter η.
- Relaxation time quantifies material 'memory'.

Problem Setup

Planar, steady flow around corner of angle $\theta = \pi/\alpha$. $(1/2 \leq \alpha < 1)$

Fluid incompressible and isothermal



No-slip and no-flux conditions on walls.

Mathematics due to Hinch (1993), Renardy (1995), Rallison and Hinch (2004) and Evans (2008).

Equations?

Dimensionless equations:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathrm{Re}\,(\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{T},$$
$$\mathrm{Wi}\,\overset{\nabla}{\boldsymbol{T}} + \boldsymbol{T} = 2\boldsymbol{D}.$$

Obtain Reynolds (Re) and Weissenberg (Wi) numbers:

$$\mathrm{Re} = \frac{\text{[inertial forces]}}{\text{[viscous forces]}}; \quad \mathrm{Wi} = \frac{\text{[elastic forces]}}{\text{[viscous forces]}}.$$

Useful: incompressibility condition gives a *streamfunction* ψ with

$$\mathbf{u} = \nabla \wedge \psi \mathbf{k}.$$

These equations are complicated!

Step 1: Outer Solution

Governing Equations

- Assume elasticity drives the flow, and inertia is subdominant.
- Find at leading order as $r \to 0$:

$$\overset{\nabla}{\boldsymbol{T}} = \mathbf{0}, \quad \nabla p = \nabla \cdot \boldsymbol{T}.$$

- First equation solved by $\boldsymbol{T} = \lambda(\psi)\mathbf{u}\mathbf{u}^T$.
- Setting $\mathbf{v} = \lambda^{1/2}\mathbf{u}$, obtain

$$(\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla p, \quad \nabla \cdot \mathbf{v} = 0.$$

- Outer solution governed by the Euler equations!

Step 1: Outer Solution

Finding the Streamfunction

$$(\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla p, \quad \nabla \cdot \mathbf{v} = 0.$$

- Incompressibility: $\mathbf{v} = \nabla \wedge \psi^* \mathbf{k}$.
- Planar flow: $\nabla^2 \psi^* = 0$.
- Applying no-slip conditions: $\psi^* \sim C r^\alpha \sin(\alpha\theta)$.
- As $\mathbf{u}$ is parallel to $\mathbf{v}$, $\psi^* = h(\psi)$, with $\lambda^{1/2}(\psi) = h'(\psi)$
- Since we are local to the corner, take

$$\psi^* \propto \psi^{1/n},$$

with n to be determined.

Step 1: Outer Solution

Summary

Consequently, near the corner apex:

$$\psi \sim C_0(r^\alpha \sin(\alpha\theta))^n,$$
$$\lambda \sim C_1 C_0^{2(\frac{1}{n}-1)}(r^\alpha \sin(\alpha\theta))^{2(1-n)},$$
$$\boldsymbol{T} \sim \lambda(\psi)\mathbf{u}\mathbf{u}^T,$$
$$p \sim \frac{1}{2}\lambda\|\mathbf{u}\|^2,$$

for unknown C_0, C_1 and n.

To neglect inertia, we also find that $n > 1$.

Are we done?

Look near the wall. In this region:

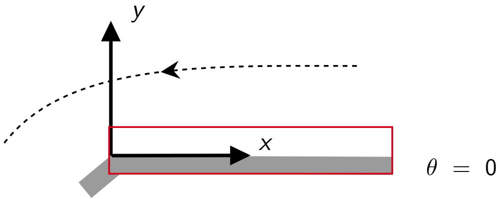
$$\psi \sim \frac{1}{2}\underbrace{\frac{\partial^2\psi}{\partial y^2}(x,0)}_{\dot{\gamma}}\,y^2 \quad \text{as} \quad y \to 0\ (x>0),$$

from which $$T_{11} \sim 2\mathrm{Wi}\dot{\gamma}^2, \quad T_{12} \sim \dot{\gamma}, \quad T_{22} = o(1).$$

However, from the outer solution, as $y \to 0$ with $x > 0$:

$$r \sim x, \quad \theta \sim y/x, \quad T_{12} \sim \text{constant}.x^{2(\alpha-1)-1}y.$$

These don't agree, so we need boundary layers.



Step 2: Inner Solution

Rescaling

To find the boundary layer equations, we only need to focus on the upstream wall.

1. Rescale (Cartesian) independent variables:

$$x = \epsilon \hat{x}, \quad y = \delta(\epsilon)\hat{y}$$

   with $0 < \epsilon \ll 1$ and $\delta \ll \epsilon$.

2. Using outer solution, rescale remaining variables:

$$\psi = \epsilon^{n(\alpha-1)}\delta^n \hat{\psi}, \quad p = \epsilon^{2(\alpha-1)}\hat{p}, \quad T_{11} = \epsilon^{2(\alpha-1)}\hat{T}_{11},$$
$$T_{12} = \epsilon^{2(\alpha-1)-1}\delta \hat{T}_{12}, \quad T_{22} = \epsilon^{2(\alpha-2)}\delta^2 \hat{T}_{22}.$$

3. Inserting into UCM ($\mathrm{Wi}\, \overset{\nabla}{\boldsymbol{T}} + \boldsymbol{T} = 2\boldsymbol{D}$), dominant balance gives

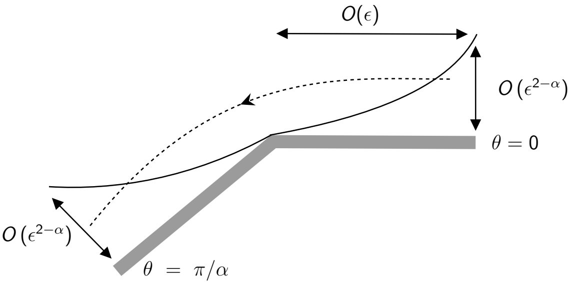
$$n = 3 - \alpha \text{ and } \delta = \epsilon^{2-\alpha}.$$

Step 2: Inner Solution

What do we learn?

1. Boundary layers of thickness $O(\epsilon^{2-\alpha})$ exist at walls.
2. Layers separated by an outer region solved for in Step 1.
3. Layers governed by leading order UCM equations and momentum equation

$$\frac{\partial \hat{T}_{11}}{\partial \hat{x}} + \frac{\partial \hat{T}_{12}}{\partial \hat{y}} = \frac{\partial \hat{p}}{\partial \hat{x}}.$$



Step 2: Inner Solution

Boundary layer UCM equations:

$$\hat{T}_{11} + \mathrm{Wi}\left[\hat{u}\frac{\partial \hat{T}_{11}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}_{11}}{\partial \hat{y}} - 2\frac{\partial \hat{u}}{\partial \hat{x}}\hat{T}_{11} - 2\frac{\partial \hat{u}}{\partial \hat{y}}\hat{T}_{12}\right] = 0,$$

$$\hat{T}_{12} + \mathrm{Wi}\left[\hat{u}\frac{\partial \hat{T}_{12}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}_{12}}{\partial \hat{y}} - \frac{\partial \hat{v}}{\partial \hat{x}}\hat{T}_{11} - \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}}\right)\hat{T}_{12} - \frac{\partial \hat{u}}{\partial \hat{y}}\hat{T}_{22}\right] = \frac{\partial \hat{u}}{\partial \hat{y}},$$

$$\hat{T}_{22} + \mathrm{Wi}\left[\hat{u}\frac{\partial \hat{T}_{22}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}_{22}}{\partial \hat{y}} - 2\frac{\partial \hat{v}}{\partial \hat{x}}\hat{T}_{12} - 2\frac{\partial \hat{u}}{\partial \hat{y}}\hat{T}_{22}\right] = 2\frac{\partial \hat{v}}{\partial \hat{y}}.$$

We can still do more with these!

Step 3: Similarity Solution

- Seek a solution in the layer of the form

$$\hat{\psi} = \hat{x}^{3-\alpha} f(\xi), \ \hat{T}_{11} = \hat{x}^{2(\alpha-1)} t_{11}(\xi), \ \hat{T}_{12} = \hat{x}^{\alpha-1} t_{12}(\xi), \ \hat{T}_{22} = t_{22}(\xi).$$

- Similarity variable:

$$\xi = \frac{\hat{y}}{\hat{x}^{2-\alpha}}$$

- Substitution into layer equations reduces the PDE system to four ODEs.
- Furthermore, we find that $\frac{\partial \hat{p}}{\partial \hat{x}} =: G$ is constant.

Step 3: Similarity Solution

UCM equations:

$$t_{11} + \mathrm{Wi}\left\{2(\alpha-2)f' t_{11} - (3-\alpha) f t_{11}' + 2(2-\alpha)\xi f'' t_{11} - 2f'' t_{12}\right\} = 0$$

$$t_{12} + \mathrm{Wi}\Big\{(\alpha-1)f' t_{12} - (3-\alpha) f t_{12}' - f'' t_{22} + (2-\alpha)(3-\alpha) f t_{11} - (2-\alpha)(3-\alpha)\xi f' t_{11} + (2-\alpha)^2 \xi^2 f'' t_{11}\Big\} = f''$$

$$t_{22} + \mathrm{Wi}\Big\{-(3-\alpha) f t_{22}' + 2(2-\alpha)(3-\alpha) f t_{12} - 2(2-\alpha)(3-\alpha) f' \xi t_{12} + 2(2-\alpha)^2 \xi^2 f'' t_{12} + 2f' t_{22} - 2(2-\alpha)\xi f'' t_{22}\Big\} = -2f' + 2(2-\alpha)\xi f''.$$

Momentum equation:

$$2(\alpha-1)t_{11} - (2-\alpha)\xi t_{11}' + t_{12}' = G.$$

Can We Solve This?

- Wall Conditions: $f(0) = 0 = f'(0)$.
- Matching: As $\xi \to \infty$

$$f \sim \pm f_0 \xi^{3-\alpha}, \; t_{11} \sim t_0,$$
$$t_{12} \sim t_0(1-\alpha)\xi, \; t_{22} \sim t_0(\alpha-1)^2\xi^2,$$

  with $f_0 := C_0\alpha^{3-\alpha}$ and $t_0 := G/(\alpha - 1)$.
- Upstream problem: formulate as IVP and integrate away from wall.
- Downstream problem: IVP unstable, need to formulate as a BVP.
- Issue: Lost stress information in the outer region — need a *natural stress formulation*!

Natural Stress Formulation?

- Realign the stress tensor basis with the flow direction:

$$\boldsymbol{T} = T_{11}\mathbf{i}\otimes\mathbf{i} + T_{12}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + T_{22}\mathbf{j}\otimes\mathbf{j}, \\ = \lambda(\psi)\mathbf{u}\otimes\mathbf{u} + \mu(\psi)(\mathbf{u}\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{u}) + \nu(\psi)\mathbf{w}\otimes\mathbf{w}.$$

- In Cartesian co-ordinates,

$$\mathbf{u} = (u, v, 0)^T,\ \mathbf{w} = \left(-\frac{v}{u^2+v^2}, \frac{u}{u^2+v^2}, 0\right)^T.$$

- Already found λ in the outer region; posit $\mu \sim C_2\psi^{n_2},\ \nu \sim C_3\psi^{n_3}$.
- In the boundary layers, we seek a similarity solution of the form

$$\lambda = x^{2(\alpha-2)}\tilde{\lambda}(\xi),\ \mu = x^{\alpha-1}\tilde{\mu}(\xi),\ \nu = x^2\tilde{\nu}(\xi),$$

Now Can We Solve This?

- We can show (in the BL):

$$\tilde{\lambda} = t_{11}(f')^2,$$
$$\tilde{\mu} = t_{12} - \frac{t_{11}}{f'}\left[-(3-\alpha)f + (2-\alpha)\xi f'\right],$$
$$\tilde{\nu} = (f')^2 t_{22} + t_{11}\left[-(3-\alpha)f + (2-\alpha)\xi f'\right]^2 - 2f' t_{12}\left[-(3-\alpha)f + (2-\alpha)\xi f'\right].$$

- Matching gives $n_2 = \frac{\alpha-1}{3-\alpha}$ and $n_3 = \frac{2}{3-\alpha}$.
- New matching conditions: As $\xi \to \infty$

$$\tilde{\lambda} \sim C_1 f_0^{2(\frac{1}{n}-1)} \xi^{2(\alpha-2)},\ \tilde{\mu} \sim \pm C_2 f_0^{n_2} \xi^{\alpha-1},\ \tilde{\nu} \sim C_3 f_0^{n_3} \xi^2.$$

- This time, we don't lose any stress information!

Summary

In this presentation, we have

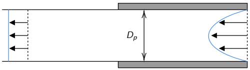
- Presented the geometry for viscoelastic re-entrant corner flow.
- Demonstrated a leading order description of the flow using matched asymptotics.
- Highlighted the benefits of using natural stress variables for solving the problem.

Where Next?

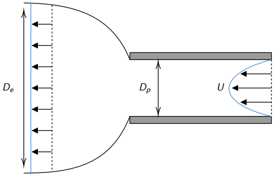
Ultimate aim: *Apply these techniques to other viscoelastic flows.*
This could be via:

1. Different viscoelastic models:
    - Including shear thinning (White-Metzner/Modified UCM)
    - Newtonian solvent inclusion.
2. Different geometries:

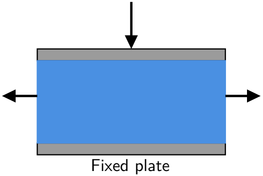
Stick-slip



Die Swell



Squeeze Flow



Thank you for listening!

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